Envy-Free Allocations for Two-Player Single-Good Allocation Problems with Money and Budget Constraints

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(Draft)

Consider the case where two players divide a single homogeneous object and are willing to transfer money up to predetermined limits. Suppose Players One and Two place monetary values of a_1 and a_2 on the object and are willing to spend up to c_1 and c_2. An allocation can be described by (x_1, x_2, m_1, m_2) where x_1 and x_2 are the fractions of the object given to players 1 and 2, and m_1 *and* m_2 are the amount of money transferred to players 1 and 2. For an allocation to be feasible, it must satisfy

$$x_1 + x_2 = 1\,, \qquad 1 \geq x_1 \geq 0\,, 1 \geq x_2 \geq 0 \tag{1}$$

$$m_1 + m_2 = 0\,, \qquad m_1 \geq -c_1\,, m_2 \geq -c_2 \tag{2}$$

Constraint (1) requires that the entire good be distributed between both players with none receiving a negative portion of it. Constraint (2) requires that money be transferred between the players and that neither may spend more than their budget limit.

We assume that Player One is the highest bidder and that each Player's budget constraint is non-negative.

$$a_1 > a_2 > 0 \tag{3}$$

$$c_1 \geq 0\,, c_2 \geq 0 \tag{4}$$

By constraints (1) and (2), the set of feasible allocations for a 2-player game with one object is a 2-dimensional rectangle in R^4, (x_1, x_2, m_1, m_2), as shown in Figure 1.

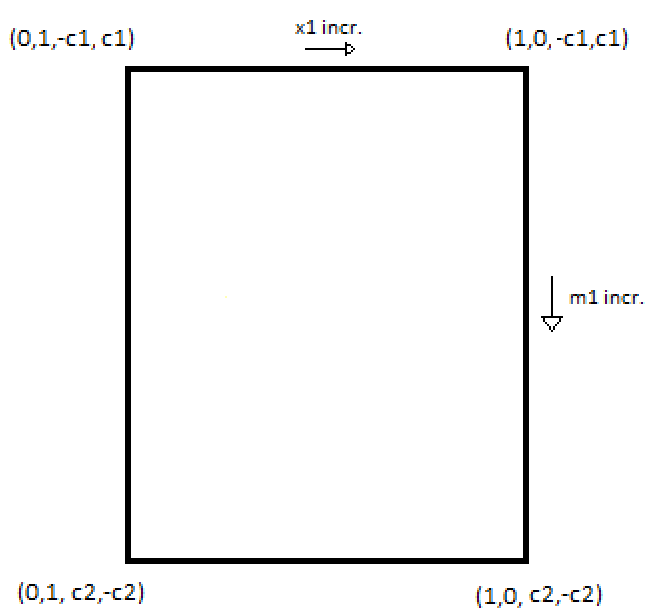


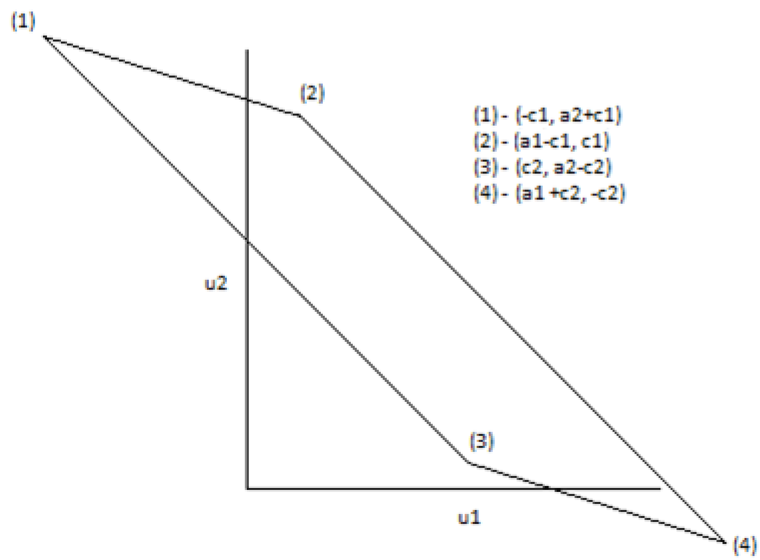
Figure 1

We assume that a player's utility for an allocation is simple the monetary value of the object multiplied by the amount of object obtained plus the amount of money obtained. Hence, an allocation in utility space, (u_1, u_2), comes directly from a linear transformation of the points in allocation space. A point (x_1, x_2, m_1, m_2) in allocation space is transformed to $(a_1x_1 + m_1, a_2x_2 + m_2)$ in utility space.

More generally,

$$u_i = a_ix_i + m_i$$

As shown, feasible allocations in Utility space are a 2-Dimensional object in R^2. The four vertices below are the transformed vertices from Figure (1).



2 SubSection 2.2.2: All Possible EnvyFree Allocations are Convex Combinations

For an allocation to be envy-free it must satisfy the feasibility constraints as well as the following inequalities. These inequalities amount to each player believing that, in their own perspective, the other Player is not receiving a greater utility value. Inequality (5) is Player Two's normal utility value compared against Player One's utility value, using Player Two's evaluation of the good. The left hand side of Inequality (7), shown below, is Player One's normal utility value while the right hand side is Player One's utility of what Player Two obtained.

$$a_2x_1 + m_1 \leq a_2x_2 + m_2 \qquad \textbf{(5)}$$

$$a_2x_1 + m_1 \leq a_2(1 - x_1) + (-m_1) \qquad \text{by constraints (1) \& (2)}$$

$$x_1 \leq \frac{1}{2} - \frac{m_1}{a_2} \qquad \textbf{(6)}$$

$$a_1x_1 + m_1 \geq a_1x_2 + m_2 \qquad \textbf{(7)}$$

$$a_1x_1 + m_1 \geq a_1(1 - x_1) + (-m_1) \qquad \text{by constraints (1) \& (2)}$$

$$x_1 \geq \tfrac{1}{2} - \tfrac{m_1}{a_1} \qquad \textbf{(8)}$$

We will define particular limits on x_1 and x_2 . To do so we will be adding Inequalities (5) and (7) :

$$\begin{array}{l} a_2x_1 + m_1 \leq a_2x_2 + m_2 \quad + \\ \underline{a_1x_2 + m_2 \leq a_1x_1 + m_1} \end{array}$$

$$a_2x_1 + a_1x_2 \leq \ a_2x_2 + a_1x_1$$

$$x_1(a_1 - a_2) \geq x_2(a_1 - a_2) \qquad \text{since } a_1 - a_2 > 0 \text{ by constraint (3)}$$

$$x_1 \geq x_2$$

$$x_1 + x_1 \geq x_2 + x_1$$

$$2x_1 \geq 1$$

$$x_1 \geq \tfrac{1}{2} \qquad \text{by equation (1),} \qquad \textbf{(9)}$$

$$x_2 \leq \tfrac{1}{2} \qquad \text{by equation (1),} \qquad \textbf{(10)}$$

Similarly for m_1 and m_2.

$$x_1 \leq \tfrac{1}{2} - \tfrac{m_1}{a_2} \qquad \text{by inequality (6)}$$

$$\tfrac{m_1}{a_2} \leq \tfrac{1}{2} - x_1$$

$$\tfrac{m_1}{a_2} \leq 0 \qquad \text{by inequality (9)}$$

$$m_1 \leq 0 \qquad \textbf{(11)}$$

$$m_2 \geq 0 \qquad \text{by constraint (2),} \qquad \textbf{(12)}$$

Geometrically, it is clear that if $c_1 \geq \frac{a_1}{2}$ the set of envy-free allocations are

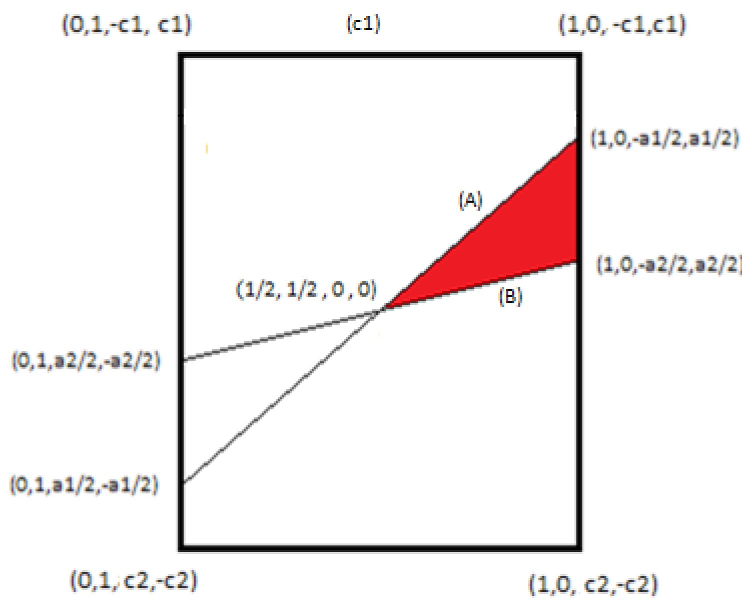
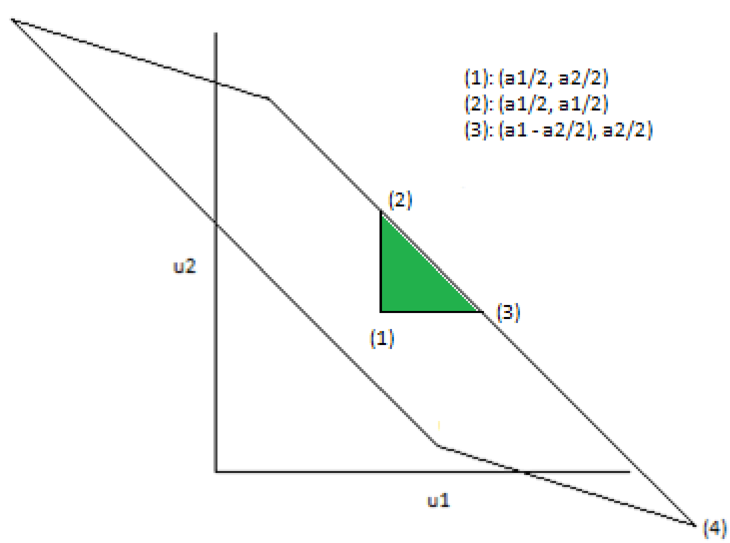


Figure (3)

Lines (A) and (B) represent the two boundaries of inequalities (7) and (5), respectively. From the graph, it is clear that the set of envy-free allocations is a convex set with vertices $V^1 = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right), V^2 = \left(1, 0, -\frac{a_1}{2}, \frac{a_1}{2}\right), and\ V^3 = \left(1, 0, -\frac{a_2}{2}, \frac{a_2}{2}\right)$.

The set of envy free allocations in utility space are shown below.



The three vertices in the Figure above are transformations of V^1, V^2 and V^3.

SubSection 2.2.2: All Possible EnvyFree Allocations are Convex Combinations

Claim 1.1: Suppose $c_1 \geq \frac{a_1}{2}$. If (x, m) is a convex combination of V^1, V^2 and V^3 as defined by the equations above, then $(x, m)$is an envy-free allocation.

Conversely we must show that if (x, m) is a convex combination of V^1, V^2, and V^3, then (x, m) is an envy free allocation. Suppose α_1, α_2, and α_3 are nonnegative numbers satisfying $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and

$(x, m) = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$

$$= \alpha_1 \left(\frac{1}{2}, \frac{1}{2}, 0, 0 \right) + \alpha_2 \left(1, 0, -\frac{a_1}{2}, \frac{a_1}{2} \right) + \alpha_3 \left(1, 0, -\frac{a_2}{2}, \frac{a_2}{2} \right)$$

That is,

$x_1 = \frac{1}{2}\alpha_1 + \alpha_2 + \alpha_3$

$x_2 = \dfrac{1}{2}\alpha_1$

$m_1 = -\frac{a_1}{2}(\alpha_2) - \frac{a_2}{2}(\alpha_3)$

$m_2 = \frac{a_1}{2}(\alpha_2) + \frac{a_2}{2}(\alpha_3)$

In order for this allocation to be envy-free it must satisfy the two inequalities, (5) and (7).

Inequality (7), Player One believes he received at least as much as Player Two, is equivalent to

$$a_1\left(\frac{1}{2}\alpha_1 + \alpha_2 + \alpha_3\right) - \left(\frac{\alpha_2 a_1 + \alpha_3 a_2}{2}\right) \geq \frac{a_1 \alpha_1}{2} + \frac{\alpha_2 a_1 + \alpha_3 a_2}{2}$$

$$\frac{\alpha_1 a_1 + \alpha_2 a_1 + \alpha_3(2a_1 - a_2)}{2} \geq \frac{\alpha_1 a_1 + \alpha_2 a_1 + \alpha_3 a_2}{2}$$

$$\alpha_3 a_1 \geq \alpha_3 a_2$$

Since $\alpha_3 \geq 0$ and $a_1 > a_2$ by inequality (3), the inequality above holds.

Inequality (5), Player Two believes he received at least as much as Player One, is equivalent to

$$a_2\left(\frac{1}{2}\alpha_1 + \alpha_2 + \alpha_3\right) - \left(\frac{\alpha_2 a_1 + \alpha_3 a_2}{2}\right) \leq \frac{a_2 \alpha_1}{2} + \frac{\alpha_2 a_1 + \alpha_3 a_2}{2}$$

$$\frac{\alpha_2 a_1 + \alpha_2(2a_2 - a_1) + \alpha_3 a_2}{2} \leq \frac{\alpha_1 a_2 + \alpha_2 a_1 + \alpha_3 a_2}{2}$$

$$\alpha_2 a_2 \leq \alpha_2 a_1$$

Since $\alpha_2 \geq 0$ and $a_1 \geq a_2$, by inequality (3), the inequality above holds.

It is also necessary to show that (x, m) is feasible, satisfying equations (1) and (2).

Equation (1), the entire good is allocated between the two players, is equivalent to

$$\tfrac{1}{2}\alpha_1 + \alpha_2 + \alpha_3 + \tfrac{1}{2}\alpha_1 = 1 \qquad \text{by (definition of convex combination)}$$

By the definition of convex combinations this equality holds.

Equation (2), the sum of the two players monetary transfers is 0, is equivalent to

$$-\frac{a_1}{2}(\alpha_2) - \frac{a_2}{2}(\alpha_3) + \frac{a_1}{2}(\alpha_2) + \frac{a_2}{2}(\alpha_3) = 0$$

Thus, If (x, m) is a convex combination of $V^1, V^2, \text{and } V^3$ then (x, m) is envy free.

Claim 1.2. Suppose $c_1 \geq \frac{a_1}{2}$. If (x, m) is an envy-free allocation, then (x, m) is a convex combination of $V^1, V^2, \text{and } V^3$ as defined by equations above **.

We verify that if (x, m) is an envy free allocation, then (x, m) is a convex combination of $V^1, V^2, \text{and } V^3$. We show that there exist nonnegative $\alpha_1, \alpha_2, \text{and } \alpha_3$ that satisfy $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $(x, m) = \alpha_1 V^1 + \alpha_2 V^2 + \alpha_3 V^3$.

Let

$$\begin{aligned}
\alpha_1 &= 2x_2 \\
\alpha_2 &= \frac{(a_2 x_2 + m_2) - (a_2 x_1 + m_1)}{a_1 - a_2} \\
\alpha_3 &= \frac{(a_1 x_1 + m_1) - (a_1 x_2 + m_2)}{a_1 - a_2}
\end{aligned}$$

We can easily show that each $\alpha_i \geq 0$. From equation (1), $x_2 \geq 0$, therefore $2x_2 \geq 0$ and so $\alpha_1 \geq 0$. Since (x, m) is envy-free, $(a_2 x_2 + m_2) - (a_2 x_1 + m_1) \geq 0$ as shown by inequality (5). We also know $a_1 - a_2 > 0$ from inequality(3). Therefore $\alpha_2 \geq 0$. Using the definition of envy-free from Inequality (7), $(a_1 x_1 + m_1) - (a_1 x_2 + m_2) > 0$. It has already been shown that $a_1 - a_2 > 0$. Hence, $\alpha_3 \geq 0$.

After some algebra and use of the feasibility conditions, we can verify that

$$\alpha_1 + \alpha_2 + \alpha_3 = \cdots$$

$$= 2x_2 + \frac{(a_2x_2+m_2)-(a_2x_1+m_1)}{a_1-a_2} + \frac{(a_1x_1+m_1)-(a_1x_2+m_2)}{a_1-a_2}$$

$$= 2x_2 + \frac{(a_1-a_2)(x_1)-(a_1-a_2)(x_2)}{a_1-a_2}$$

$$= x_1 + x_2$$

$$= 1$$ by Equation (1)

Also, we must show that $\alpha_1 V^1 + \alpha_2 V^2 + \alpha_3 V^3 = (x, m)$

That is,

$$2x_2\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right) + \frac{(a_2x_2+m_2)-(a_2x_1+m_1)}{a_1-a_2}\left(1, 0, -\frac{a_1}{2}, \frac{a_1}{2}\right) + \frac{(a_1x_1+m_1)-(a_1x_2+m_2)}{a_1-a_2}\left(1, 0, -\frac{a_2}{2}, \frac{a_2}{2}\right) = (x_1, x_2, m_1, m_2)$$

The necessary calculations to verify the four component equations follow.

$$\frac{1}{2}(2x_2) + \frac{(a_2x_2+m_2)-(a_2x_1+m_1)}{a_1-a_2} + \frac{(a_1x_1+m_1)-(a_1x_2+m_2)}{a_1-a_2} = \cdots$$

$$= x_2 + \frac{(a_1-a_2)(x_1)-(a_1-a_2)(x_2)}{a_1-a_2}$$

$$= x_1$$

$$\frac{1}{2}(2x_2) = x_2$$

$$\frac{(a_2x_2+m_2)-(a_2x_1+m_1)}{a_1-a_2}\left(-\frac{a_1}{2}\right) + \frac{(a_1x_1+m_1)-(a_1x_2+m_2)}{a_1-a_2}\left(-\frac{a_2}{2}\right) = \cdots$$

$$= \frac{m_1(a_1 - a_2) - m_2(a_1 - a_2)}{2(a_1-a_2)}$$

$$= \frac{2m_1(a_1 - a_2)}{2(a_1-a_2)}$$

$$= m_1$$

$$\frac{(a_2x_2+m_2)-(a_2x_1+m_1)}{a_1-a_2}\left(\frac{a_1}{2}\right)+\frac{(a_1x_1+m_1)-(a_1x_2+m_2)}{a_1-a_2}\left(\frac{a_2}{2}\right)=\cdots$$

$$=\frac{m_2(a_1-\ a_2)-m_1(a_1-a_2)}{2(a_1-a_2)}$$

$$=\frac{2m_2(a_1-\ a_2)}{2(a_1-a_2)}$$

$$=m_2$$

So $\alpha_1V^1+\alpha_2V^2+\alpha_3V^3=(x,m)=(x_1,x_2,m_1,m_2)$.

Thus if (x,m) is an envy free allocation, then (x,m) is a convex combination of $V^1, V^2,$ and V^3.

Claim 1.3. Suppose $c_1\geq\frac{a_1}{2}$. The allocations V^1, V^2 and V^3 as defined by equations (??) are vertices of the set of envy-free allocations.

We now verify that V^1, V^2 and V^3 are vertices of the envy free allocations. We will do so by showing that if the average of any two points in the set of envy free allocations equals this allocation then these two points are in fact the same, or, the allocation itself.

Suppose (x_1,x_2,m_1,m_2) and (y_1,y_2,n_1,n_2) are envy free and

$$\frac{1}{2}(x_1,x_2,m_1,m_2)+\frac{1}{2}(y_1,y_2,n_1,n_2)=\left(\frac{1}{2},\frac{1}{2},0,0\right)$$

That is,

$$\frac{1}{2}x_1+\frac{1}{2}y_1=\frac{1}{2}$$

$$\frac{1}{2}x_2+\frac{1}{2}y_2=\frac{1}{2}$$

$$\frac{1}{2}m_1+\frac{1}{2}n_1=0$$

$$\frac{1}{2}m_2+\frac{1}{2}n_2=0$$

We know $x_1$and y_1 must be at least $\frac{1}{2}$, by inequality (9), therefore for the first component equality to hold $x_1 = y_1 = \frac{1}{2}$ because if $x_1 > \frac{1}{2}$ or $y_1 > \frac{1}{2}$ it would be necessary for $y_1 < \frac{1}{2}$ or $x_1 < \frac{1}{2}$, respectively, a contradiction. So from x_1, y_1 we know $x_2 = y_2 = \frac{1}{2}$ using equation (1) . Furthermore, $m_1 = n_1 = 0$ since if $m_1 < 0$ or $n_1 < 0$ then for equation (2) to hold it would be necessary for $n_1 > 0$ or $m_1 > 0$, respectively, a contradiction by inequality (11). Finally, $m_2 = n_2 = 0$ using the corresponding values of m_1 and n_1 and equation (2). Therefore $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ is a vertex since the points are identical.

Suppose (x_1, x_2, m_1, m_2) and (y_1, y_2, n_1, n_2) are envy free and

$$\frac{1}{2}(x_1, x_2, m_1, m_2) + \frac{1}{2}(y_1, y_2, n, n_2) = (1, 0, -\frac{a_1}{2}, \frac{a_1}{2})$$

That is,

$$\frac{1}{2}x_1 + \frac{1}{2}y_1 = 1$$

$$\frac{1}{2}x_2 + \frac{1}{2}y_2 = 0$$

$$\frac{1}{2}m_1 + \frac{1}{2}n_1 = -\frac{a_1}{2}$$

$$\frac{1}{2}m_2 + \frac{1}{2}n_2 = \frac{a_1}{2}$$

Starting with the first component equality if $x_1 < 1$ or $y_1 < 1$ it would require that if $y_1 > 1$ or $x_1 > 1$, respectively, a contradiction to constraint (1) . So $x_1 = y_1 = 1$. Then, $x_2 = y_2 = 0$ using equation(1). Starting with inequality (8) , $x_1 \geq \frac{1}{2} - \frac{m_1}{a_1}$, and substituting $x_1 = 1$ we get $m_1 \geq -\frac{a_1}{2}$ and $n_1 \geq -\frac{a_1}{2}$. If $m_1 > -\frac{a_1}{2}$ or $n_1 > -\frac{a_1}{2}$ then for the third component equality to hold either $n_1 < -\frac{a_1}{2}$ or $m_1 < -\frac{a_1}{2}$ respectively. By this, $m_1 = n_1 = -\frac{a_1}{2}$. Finally, $m_2 = n_2 = \frac{a_1}{2}$ using the corresponding values of m_1 and n_1 and using equation (2) . Therefore $(1, 0, -\frac{a_1}{2}, \frac{a_1}{2})$ is a vertex.

Suppose(x_1, x_2, m_1, m_2) and (y_1, y_2, n_1, n_2) are envy free and

$$\frac{1}{2}(x_1, x_2, m_1, m_2) + \frac{1}{2}(y_1, y_2, n_1, n_2) = (1, 0, -\frac{a_2}{2}, \frac{a_2}{2})$$

That is,

$$\frac{1}{2}x_1 + \frac{1}{2}y_1 = 1$$

$$\frac{1}{2}x_2 + \frac{1}{2}y_2 = 0$$

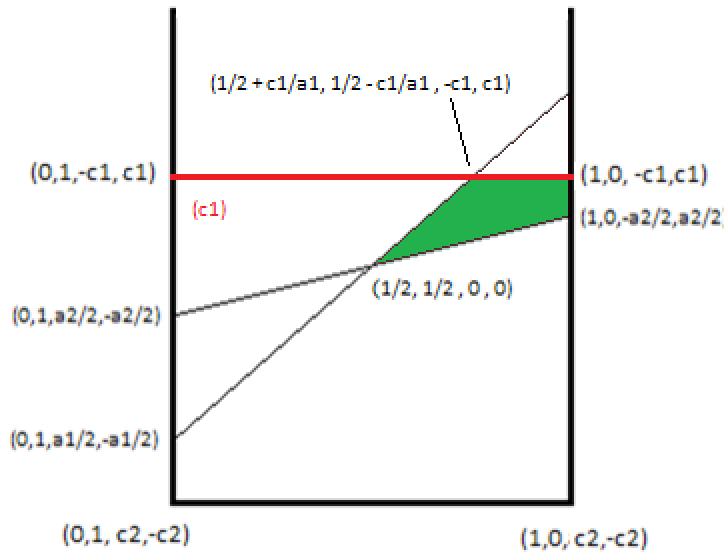
$$\frac{1}{2}m_1 + \frac{1}{2}n_1 = -\frac{a_2}{2}$$

$$\frac{1}{2}m_2 + \frac{1}{2}n_2 = \frac{a_2}{2}$$

Starting with the first equality, if $x_1 < 1$ or $y_1 < 1$ it would be necessary for either $y_1 > 1$ or $x_1 > 1$, respectively, a contradiction to constraint (1) . So $x_1 = y_1 = 1$. Then, $x_2 = y_2 = 0$ using equation(1). From inequality (6) and substituting $x_1 = 1$ we get $m_1 \leq -\frac{a_2}{2}$ and $n_1 \leq -\frac{a_2}{2}$. In the third component equality if $m_1 < -\frac{a_2}{2}$ or $n_1 < -\frac{a_2}{2}$ then it would require that $m_1 > -\frac{a_2}{2}$ or $n_1 > -\frac{a_2}{2}$, respectively, a contradiction to inequality (6). Therefore $m_1 = n_1 = -\frac{a_2}{2}$. Finally, $m_2 = n_2 = \frac{a_1}{2}$ using the corresponding values of m_1 and n_1 and using equation (2). Hence $\left(1, 0, -\frac{a_2}{2}, \frac{a_2}{2} \right)$ is a vertex.

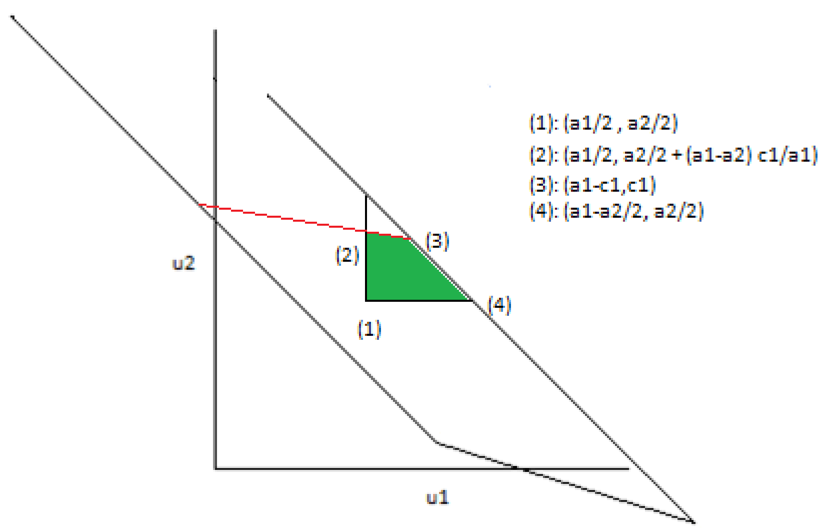
SubSection 2.2.2: All Possible EnvyFree Allocations are Convex Combinations

If , $\frac{a_1}{2} > c_1 > \frac{a_2}{2}$, the set of envy free allocations forms a different shape set inside the rectangular set of feasible allocations.



Figure

And the utility space is



The entire set of vertices in allocation space:

Table 3	(x_1, x_2, m_1, m_2)
V^1	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$
V^2	$(\frac{1}{2} + \frac{c_1}{a_1}, \frac{1}{2} - \frac{c_1}{a_1}, -c_1, c_1)$
V^3	$(1, 0, -c_1, c_1)$
V^4	$(1, 0, -\frac{a_2}{2}, \frac{a_2}{2})$

Claim 2.1. Suppose $\frac{a_1}{2} > c_1 > \frac{a_2}{2}$. If $(x, m)$is a convex combination of $V^1, V^2, V^3,$ and V^4 as defined by Table 3 above, then $(x, m)$is an envy-free allocation.

Therefore if (x, m) is an envy free allocation it can be expressed as a convex combination of V^1, V^2, V^3, V^4

We must show that if (x, m) is an allocation that is a convex combination of the vertices above then it must be envy-free.

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \alpha_4 V_4 = \cdots$$

$$= \alpha_1 \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right) + \alpha_2 \left(\frac{1}{2} + \frac{c_1}{a_1}, \frac{1}{2} - \frac{c_1}{a_1}, -c_1, c_1\right) + \alpha_3 (1, 0, -c_1, c_1) + \alpha_4 (1, 0, -\frac{a_2}{2}, \frac{a_2}{2})$$

So,

$$x_1 = \frac{1}{2}\alpha_1 + \left(\frac{1}{2} + \frac{c_1}{a_1}\right)\alpha_2 + \alpha_3 + \alpha_4$$

$$x_2 = \frac{1}{2}\alpha_1 + \left(\frac{1}{2} - \frac{c_1}{a_1}\right)\alpha_2$$

$$m_1 = -c_1\alpha_2 - c_1\alpha_3 - \frac{a_2\alpha_4}{2}$$

$$m_2 = c_1\alpha_2 + c_1\alpha_3 + \frac{a_2\alpha_4}{2}$$

In order for this allocation to be envy-free it must satisfy the two inequalities, (5) and (7).

Inequality (7), Player One believes he received at least as much as Player Two, is equivalent to

$$\left(\frac{1}{2}\alpha_1 + \left(\frac{1}{2}+\frac{c_1}{a_1}\right)\alpha_2 + \alpha_3 + \alpha_4\right)a_1 + \left(-c_1\alpha_2 - c_1\,\alpha_3 - \frac{a_2\alpha_4}{2}\right) \geq \left(\frac{1}{2}\alpha_1 + \left(\frac{1}{2}-\frac{c_1}{a_1}\right)\alpha_2\right)a_1 + c_1\alpha_2 + c_1\,\alpha_3 + \frac{a_2\alpha_4}{2}$$

$$\frac{1}{2}\alpha_1 a_1 + \frac{1}{2}\alpha_2 a_1 + (a_1 - c_1)\alpha_3 + (a_1 - \frac{a_2}{2})\alpha_4 \geq \frac{1}{2}\alpha_1 a_1 + \frac{1}{2}\alpha_2 a_1 + c_1\,\alpha_3 + \frac{a_2\alpha_4}{2}$$

$$(a_1 - 2c_1)\alpha_3 + (a_1 - a_2)\alpha_4 \geq 0$$

From Claim 2.1, $2c_1 < a_1$, and $a_1 - a_2 > 0$ by constraint (3).

Inequality (5), Player Two believes he received at least as much as Player One, is equivalent to

$$\left(\frac{1}{2}\alpha_1 + \left(\frac{1}{2}+\frac{c_1}{a_1}\right)\alpha_2 + \alpha_3 + \alpha_4\right)a_2 + \left(-c_1\alpha_2 - c_1\,\alpha_3 - \frac{a_2\alpha_4}{2}\right) \leq \left(\frac{1}{2}\alpha_1 + \left(\frac{1}{2}-\frac{c_1}{a_1}\right)\alpha_2\right)a_2 + c_1\alpha_2 + c_1\,\alpha_3 + \frac{a_2\alpha_4}{2}$$

$$\frac{1}{2}\alpha_1 a_2 + \left(\frac{1}{2}a_2 + \frac{c_1(a_2 - a_1)}{a_1}\right)\alpha_2 + (a_2 - c_1)\alpha_3 + \frac{a_2\alpha_4}{2} \leq \frac{1}{2}\alpha_1 a_2 + \left(\frac{1}{2}a_2 + \frac{c_1(a_1 - a_2)}{a_1}\right)\alpha_2 + c_1\,\alpha_3 + \frac{a_2\alpha_4}{2}$$

$$0 \leq (2c_1 - a_2)\alpha_3 + \frac{2c_1(a_1 - a_2)}{a_1}\,\alpha_2$$

Since $\alpha_3 \geq 0$ and $\alpha_2 \geq 0$, $2c_1 - a_2 > 0$ by Claim 2.1, and $a_1 - a_2 > 0$ by constraint (3), the inequality holds.

Thus, if (x, m) is a convex combination of the vertices above then (x, m) will be envy-free.

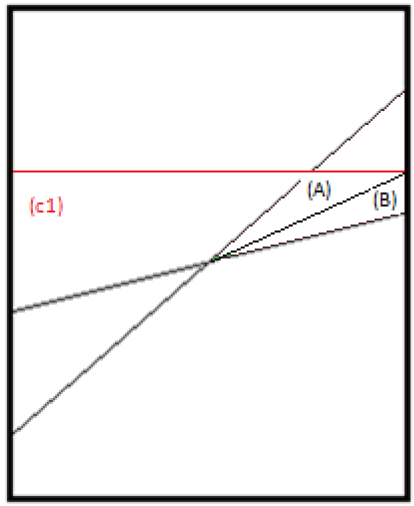
Claim 2.2. Suppose $\frac{a_1}{2} > c_1 > \frac{a_2}{2}$. If (x, m) is an envy-free allocation, then (x, m) is a convex combination of $V^1, V^2, V^3,$ and V^4 as defined by Table 3. [Note that your argument on page 15 for alpha_2 >= 0 is incorrect.

We verify that if (x, m) is an envy free allocation, then (x, m) is a convex combination of $V^1, V^2, V^3,$ and V^4. We show that there exist nonnegative $\alpha_1, \alpha_2, \alpha_3,$ and α_4 that satisfy $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$ and $(x, m) = \alpha_1 V^1 + \alpha_2 V^2 + \alpha_3 V^3 + \alpha_4 V^4$

In order to simplify this case we will look at two separate cases of the envy-free region.

$2c_1x_1 + m_1 \leq c_1$ and $2c_1x_1 + m_1 \geq c_1$

Graphically the two spaces are represented by the two regions, A and B.



Case (A): $\mathbf{2c_1x_1 + m_1 \leq c_1}$

	(x_1, x_2, m_1, m_2)
V^1	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$
V^2	$(\frac{1}{2} + \frac{c_1}{a_1}, \frac{1}{2} - \frac{c_1}{a_1}, -c_1, c_1)$
V^3	$(1, 0, -c_1, c_1)$

Let

$$\alpha_1 = \frac{c_1 + m_1}{c_1}$$
$$\alpha_2 = \frac{a_1(c_1x_2 - c_1x_1 - m_1)}{c_1(a_1 - 2c_1)}$$
$$\alpha_3 = \frac{(a_1x_1 + m_1) - (a_1x_2 + m_2)}{a_1 - 2c_1}$$

$$\alpha_4 = 0$$

From Inequality (2), $m_1 \geq -c_1$, we know $c_1 + m_1 \geq 0$. So $\alpha_1 \geq 0$. Using the constraints of this case, $\frac{a_1}{2} > c_1 > \frac{a_2}{2}$, so $c_1(a_1 - 2c_1) \geq 0$. In order for $c_1x_1 - c_1x_2 - m_1 \geq 0$ then $2c_1x_1 + m_1 \leq c_1$, which from the definition of Region A, holds. Therefore $\alpha_2 \geq 0$. $(a_1x_1 + m_1) - (a_1x_2 + m_2) \geq 0$ follows from Inequality (7) and envy free thus $\alpha_3 \geq 0$.

$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \cdots$

$$= \frac{c_1 + m_1}{c_1} + \frac{a_1(c_1x_1 - c_1x_2 - m_1)}{c_1(a_1 - 2c_1)} + \frac{(a_1x_1 + m_1) - (a_1x_2 + m_2)}{a_1 - 2c_1}$$

$$= \frac{c_1 + m_1}{c_1} + \frac{-a_1c_1x_1 + a_1c_1x_2 + a_1c_1x_1 - a_1c_1x_2 - (a_1 - 2c_1)(m_1)}{c_1(a_1 - 2c_1)}$$

$$= \frac{c_1 + m_1}{c_1} - \frac{m_1}{c_1}$$
$$= 1$$

Also, we must show that $\alpha_1V^1 + \alpha_2V^2 + \alpha_3V^3 + \alpha_4V^4 = (x, m)$

$$\left(\frac{c_1 + m_1}{c_1}\right)\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right) + \left(\frac{a_1(c_1x_2 - c_1x_1 - m_1)}{c_1(a_1 - 2c_1)}\right)\left(\frac{1}{2} + \frac{c_1}{a_1}, \frac{1}{2} - \frac{c_1}{a_1}, -c_1, c_1\right) + \frac{(a_1x_1 + m_1) - (a_1x_2 + m_2)}{a_1 - 2c_1}(1, 0, -c_1, c_1) + (0)\left(1, 0, -\frac{a_2}{2}, \frac{a_2}{2}\right) = (x, m)$$

The four components of this equation are as follows

$$\frac{1}{2}\left(\frac{c_1 + m_1}{c_1}\right) + \left(\frac{1}{2} + \frac{c_1}{a_1}\right)\left(\frac{a_1\left(c_1x_2 - c_1x_1 - m_1\right)}{c_1\left(a_1 - 2c_1\right)}\right) + \frac{(a_1x_1 + m_1) - (a_1x_2 + m_2)}{a_1 - 2c_1} =$$

$$= \left(\frac{c_1 + m_1}{2c_1}\right) + \frac{a_1(c_1x_2 - c_1x_1 - m_1)}{2c_1(a_1 - 2c_1)} + \frac{(a_1 - c_1)x_1 - (a_1 - c_1)x_2 - m_1}{a_1 - 2c_1}$$

$$= \left(\frac{c_1 + m_1}{2c_1}\right) + \frac{(a_1 - 2c_1)c_1x_1 - (a_1 - 2c_1)c_1x_2 - (a_1 - 2c_1)m_1}{2c_1(a_1 - 2c_1)}$$

$$= \frac{c_1 + c_1\left(x_1 - (1 - x_1)\right)}{2c_1}$$

$$= \frac{2c_1x_1}{2c_1}$$

$= x_1$

$$\frac{1}{2}\left(\frac{c_1+m_1}{c_1}\right)+\left(\frac{1}{2}-\frac{c_1}{a_1}\right)\left(\frac{a_1\left(c_1x_2-c_1x_1-m_1\right)}{c_1\left(a_1-2c_1\right)}\right)=$$

$$=\left(\frac{c_1+m_1}{2c_1}\right)+\frac{x_2c_1(a_1-2c_1)-c_1x_1(a_1-2c_1)-(a_1-2c_1)m_1}{2c_1(a_1-2c_1)}$$

$$=\frac{c_1+c_1\left(x_2-(1-x_2)\right)}{2c_1}$$

$$=\frac{2c_1x_2}{2c_1}$$

$= x_2$

$$(c_1)\left(\frac{a_1(c_1x_1-c_1x_2+m_1)}{c_1(a_1-2c_1)}\right)+(c_1)\frac{(a_1x_2+m_2)-(a_1x_1+m_1)}{a_1-2c_1}=$$

$$=\frac{(a_1c_1x_1-a_1c_1x_2-a_1c_1x_1+a_1c_1x_2+(a_1-2c_1)m_1)}{a_1-2c_1}$$

$= m_1$

$$(c_1)\left(\frac{a_1(c_1x_2-c_1x_1-m_1)}{c_1(a_1-2c_1)}\right)+(c_1)\frac{(a_1x_1+m_1)-(a_1x_2+m_2)}{a_1-2c_1}=$$

$$=\frac{(a_1c_1x_1-a_1c_1x_2-a_1c_1x_1+a_1c_1x_2+(a_1-2c_1)m_2)}{a_1-2c_1}$$

$= m_2$

Indeed then $\alpha_1V^1+\alpha_2V^2+\alpha_3V^3+\ \alpha_4V^4=(x,m)=\ (x_1,x_2,m_1,m_2)$

Case(B): $\boldsymbol{2c_1x_1+m_1\geq c_1}$

Let

$$\alpha_1 = 2x_2$$
$$\alpha_2 = 0$$
$$\alpha_3=\frac{(a_2x_1+m_1)-(a_2x_2+m_2)}{a_2-2c_1}$$

$$\alpha_4 = \frac{2(c_1x_2 - c_1x_1 - m_1)}{a_2 - 2c_1}$$

By Equation (1) $\alpha_1 \geq 0$. Since(x, m) is an envy-free allocation $(a_2x_1 + m_1) - (a_2x_2 + m_2) \leq 0$ Inequality (5). $2c_1 > a_2$ by the stated constraint of this case, $\frac{a_1}{2} > c_1 > \frac{a_2}{2}$. . Therefore $\alpha_3 \geq 0$. According to Inequalities (6) and (8), $x_1 \leq x_2 - \frac{m_1}{c_1}$ because $x_2 \leq \frac{1}{2}$ by Inequality (1.0) and $a_2 > c_1 < a_1$ and so $\alpha_4 \geq 0$.

Using algebra, we can verify that

$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \cdots$

$$= 2x_2 + \frac{(a_2x_1 + m_1) - (a_2x_2 + m_2)}{a_2 - 2c_1} + \frac{2(c_1x_2 - c_1x_1 - m_1)}{a_2 - 2c_1}$$

$$= 2x_2 + \frac{(a_2 - 2c_1)(x_1)) - (a_2 - 2c_1)(x_2)}{a_2 - 2c_1}$$

$$= x_1 + x_2$$

$$= 1 \qquad \text{by Equation (1)}$$

Also, we must show that $\alpha_1 V^1 + \alpha_2 V^2 + \alpha_3 V^3 + \alpha_4 V^4 = (x, m)$

$$2x_2(\frac{1}{2}, \frac{1}{2}, 0, 0) + (0)(\frac{1}{2} + \frac{c_1}{a_1}, \frac{1}{2} - \frac{c_1}{a_1}, -c_1, c_1) + \frac{(a_2x_1 + m_1) - (a_2x_2 + m_2)}{a_2 - 2c_1}(1, 0, -c_1, c_1) + \frac{2(c_1x_2 - c_1x_1 - m_1)}{a_2 - 2c_1}\left(1, 0, -\frac{a_2}{2}, \frac{a_2}{2}\right) = (x, m)$$

Breaking this down into four equality components we get

$$\frac{1}{2}(2x_2) + \frac{(a_2x_1 + m_1) - (a_2x_2 + m_2)}{a_2 - 2c_1} + \frac{2(c_1x_2 - c_1x_1 - m_1)}{a_2 - 2c_1} = \cdots$$

$$= x_2 + \frac{(a_2 - 2c_1)(x_1)) - (a_2 - 2c_1)(x_2)}{a_2 - 2c_1}$$

$$= x_1 \qquad \text{from the computation of } \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$\frac{1}{2}(2x_2) = x_2$$

$$\frac{(a_2x_1+m_1)-(a_2x_2+m_2)}{a_2-2c_1}(-c_1) + \frac{2(c_1x_2-c_1x_1-m_1)}{a_2-2c_1}\left(-\frac{a_2}{2}\right) = \cdots$$

$$= \frac{c_1m_2 - c_1m_1 + a_2m_1}{a_2-2c_1}$$

$$= \frac{a_2-2c_1}{a_2-2c_1}(m_1)$$ by Equation(2)

$$= m_1$$

$$\frac{(a_2x_1+m_1)-(a_2x_2+m_2)}{a_2-2c_1}(c_1) + \frac{2(c_1x_2-c_1x_1-m_1)}{a_2-2c_1}\left(\frac{a_2}{2}\right) = \cdots$$

$$= \frac{-c_1m_2 + c_1m_1 - a_2m_1}{a_2-2c_1}$$

$$= \frac{a_2-2c_1}{a_2-2c_1}(m_2)$$ by Equation(2)

$$= m_2$$

Indeed then $\alpha_1V^1 + \alpha_2V^2 + \alpha_3V^3 = (x, m) = (x_1, x_2, m_1, m_2)$.

Claim 2.3. Suppose $\frac{a_1}{2} > c_1 > \frac{a_2}{2}$. The allocations V^1, V^2, V^3, and V^4 as defined by Table 3 are vertices of the set of envy-free allocations.

We now verify that V^1, V^2, V^3 and V^4 are vertices of the envy free allocations. We will do so by showing that if the average of any two points in the set of envy free allocations equals this allocation then these two points are in fact the same, or, the allocation itself.

$$\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$$

As shown in the previous case we know that this is a vertex of the set.

Suppose (x_1, x_2, m_1, m_2) and (y_1, y_2, n_1, n_2) are envy free and

$$\frac{1}{2}(x_1, x_2, m_1, m_2) + \frac{1}{2}(y_1, y_2, n_1, n_2) = \left(\frac{1}{2} + \frac{c_1}{a_1}, \frac{1}{2} - \frac{c_1}{a_1}, -c_1, c_1\right)$$

That is,

$$\frac{1}{2}x_1 + \frac{1}{2}y_1 = \frac{1}{2} + \frac{c_1}{a_1}$$

$$\frac{1}{2}x_2 + \frac{1}{2}y_2 = \frac{1}{2} - \frac{c_1}{a_1}$$

$$\frac{1}{2}m_1 + \frac{1}{2}z_1 = -c_1$$

$$\frac{1}{2}m_2 + \frac{1}{2}z_2 = c_1$$

Let's assume that $m_1 = \frac{-c_1}{q}$ where $q > 1$. Solving the third equality we then, $n_1 = -2c_1 + \frac{c_1}{q} \leq -c_1$ Therefore, if m_1 does not equal $-c_1$ then Equation (2) is violated. Hence, $m_1 = n_1 = -c_1$. Subsequently by Equation (2), $m_2 = n_2 = c_1$. Knowing this, we could instead write the first equality as $\frac{1}{2}x_1 + \frac{1}{2}y_1 = \frac{1}{2} - \frac{m_1}{a_1}$. Indeed, if $x_1 > \frac{1}{2} - \frac{m_1}{a_1}$ or $y_1 > \frac{1}{2} - \frac{m_1}{a_1}$ then it would be necessary for either $y_1 < \frac{1}{2} - \frac{m_1}{a_1}$ or $x_1 < \frac{1}{2} - \frac{m_1}{a_1}$ respectively which violates Inequality (8). Therefore, $x_1 = y_1 = \frac{1}{2} + \frac{c_1}{a_1}$. Which results in , $x_2 = y_2 = 1 - \left(\frac{1}{2} + \frac{c_1}{a_1}\right) = \frac{1}{2} - \frac{c_1}{a_1}$ from Equation(1). Following from these results $\left(\frac{1}{2} + \frac{c_1}{a_1}, \frac{1}{2} - \frac{c_1}{a_1}, -c_1, c_1\right)$ must be a vertex.

Suppose (x_1, x_2, m_1, m_2) and (y_1, y_2, n_1, n_2) are envy free and

$$\frac{1}{2}(x_1, x_2, m_1, m_2) + \frac{1}{2}(y_1, y_2, n_1, n_2) = (1, 0, -c_1, c_1)$$

That is

$$\frac{1}{2}x_1 + \frac{1}{2}y_1 = 1$$

$$\frac{1}{2}x_2 + \frac{1}{2}y_2 = 0$$

$$\frac{1}{2}m_1 + \frac{1}{2}z_1 = -c_1$$

$$\frac{1}{2}m_2 + \frac{1}{2}z_2 = c_1$$

We know from Equation (1) that neither x_2 or y_2 can be less than 0. Therefore for the second equality to hold, $x_2 = y_2 = 0$ otherwise it would be impossible for their sum to equal 0. Following from that then, $x_1 = y_1 = 1$ by Equation (1). Using the same argument above in the previous vertex we know that if $\frac{1}{2}m_1 + \frac{1}{2}z_1 = -c_1$ then $m_1 = z_1 = -c_1$. By Equation(2) then, $m_2 = z_2 = c_1$.
$(1, 0, -c_1, c_1)$ is a vertex.

$$(1, 0, -\frac{a_2}{2}, \frac{a_2}{2})$$

This is a vertex as shown in the first case.

3.1:``

The final scenario for this proof is where $0 \le c_1 \le \frac{a_2}{2}$.

The graphical representation of the allocation space is shown below.

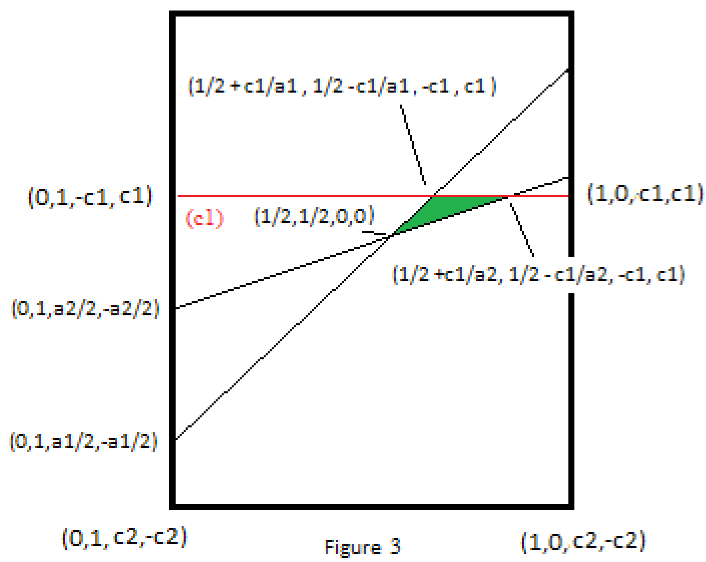


Figure 3

Utility space is

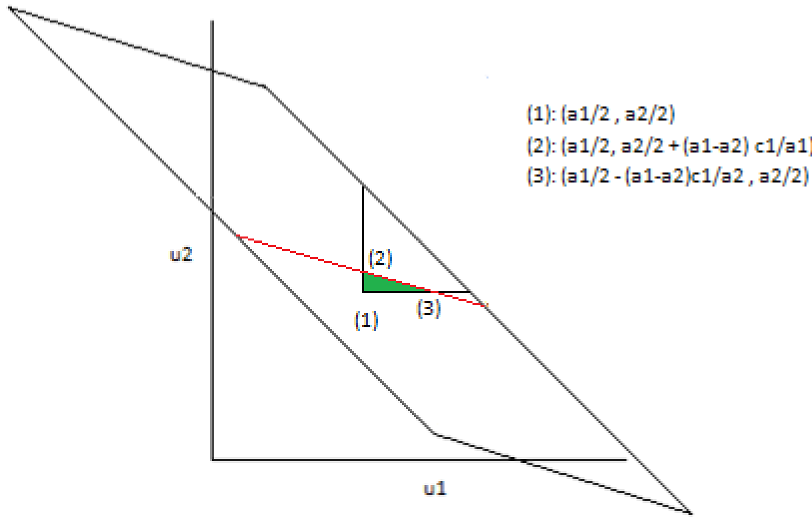


Table 4	(x_1, x_2, m_1, m_2)
V^1	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$
V^2	$(\frac{1}{2} + \frac{c_1}{a_1}, \frac{1}{2} - \frac{c_1}{a_1}, -c_1, c_1)$
V^3	$(\frac{1}{2} + \frac{c_1}{a_2}, \frac{1}{2} - \frac{c_1}{a_2}, -c_1, c_1)$

Claim 3.1: Suppose $0 \leq c_1 \leq \frac{a_2}{2}$. If (x, m) is a convex combination of $V^1, V^2,$ and V^3 as defined by Table 4, then (x, m) is an envy-free allocation.

We must verify that if (x, m) is a convex combination of $V^1, V^2,$ and V^3, then (x, m) is an envy free allocation. Suppose $\alpha_1, \alpha_2,$ and α_3 are nonnegative numbers satisfying $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and

That is,

$$(x, m) = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$$

$$= \alpha_1 \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right) + \alpha_2 \left(\frac{1}{2} + \frac{c_1}{a_1}, \frac{1}{2} - \frac{c_1}{a_1}, -c_1, c_1\right) + \alpha_3 \left(\frac{1}{2} + \frac{c_1}{a_2}, \frac{1}{2} - \frac{c_1}{a_2}, -c_1, c_1\right)$$

$$x_1 = \frac{1}{2}\alpha_1 + \left(\frac{1}{2} + \frac{c_1}{a_1}\right)\alpha_2 + \left(\frac{1}{2} + \frac{c_1}{a_2}\right)\alpha_3$$

$$x_2 = \frac{1}{2}\alpha_1 + \left(\frac{1}{2} - \frac{c_1}{a_1}\right)\alpha_2 + \left(\frac{1}{2} - \frac{c_1}{a_2}\right)\alpha_3$$

$$m_1 = -c_1\alpha_2 - c_1\alpha_3$$

$$m_2 = c_1\alpha_2 + c_1\alpha_3$$

This allocation must satisfy the Inequalities (5) and (7) below.

$$\left(\frac{1}{2}\alpha_1 + \left(\frac{1}{2} + \frac{c_1}{a_1}\right)\alpha_2 + \left(\frac{1}{2} + \frac{c_1}{a_2}\right)\alpha_3\right)a_1 + (-c_1\alpha_2 - c_1\alpha_3) \geq \left(\frac{1}{2}\alpha_1 + \left(\frac{1}{2} - \frac{c_1}{a_1}\right)\alpha_2 + \left(\frac{1}{2} - \frac{c_1}{a_2}\right)\alpha_3\right)a_1 + c_1\alpha_2 + c_1\alpha_3$$

$$\frac{1}{2}\alpha_1 a_1 + \frac{1}{2}\alpha_2 a_1 + \left(\frac{1}{2}a_1 + \frac{c_1(a_1 - a_2)}{a_2}\right)\alpha_3 \geq \frac{1}{2}\alpha_1 a_1 + \frac{1}{2}\alpha_2 a_1 + \left(\frac{1}{2}a_1 + \frac{c_1(a_2 - a_1)}{a_2}\right)\alpha_3$$

Following from Inequality (3) , $a_1 - a_2 \geq a_2 - a_1$. So this first inequality holds.

$$\left(\frac{1}{2}\alpha_1 + \left(\frac{1}{2} + \frac{c_1}{a_1}\right)\alpha_2 + \left(\frac{1}{2} + \frac{c_1}{a_2}\right)\alpha_3\right)a_2 + (-c_1\alpha_2 - c_1\alpha_3) \leq \left(\frac{1}{2}\alpha_1 + \left(\frac{1}{2} - \frac{c_1}{a_1}\right)\alpha_2 + \left(\frac{1}{2} - \frac{c_1}{a_2}\right)\alpha_3\right)a_2 + c_1\alpha_2 + c_1\alpha_3$$

$$\frac{1}{2}\alpha_1 a_2 + \left(\frac{1}{2}a_2 + \frac{c_1(a_2 - a_1)}{a_1}\right)\alpha_2 + \frac{1}{2}\alpha_3 a_2 \leq \frac{1}{2}\alpha_1 a_2 + \left(\frac{1}{2}a_2 + \frac{c_1(a_1 - a_2)}{a_1}\right)\alpha_2 + \frac{1}{2}\alpha_3 a_2$$

$a_2 - a_1 \leq a_1 - a_2$ follows from Inequality (3) so the second inequality holds.

Hence, if (x, m) is a convex combination of $V^1, V^2,$ and V^3 then it itself is an envy-free allocation.

Claim 3.2: Suppose $0 \leq c_1 \leq \frac{a_2}{2}$. If (x, m) is an envy free allocation, then (x, m) is a convex combination of $V^1, V^2,$ and V^3 as defined by Table 4.

We verify that if (x, m) is an envy free allocation, then (x, m) is a convex combination of $V^1, V^2,$ and V^3. We show that there exist nonnegative $\alpha_1, \alpha_2,$ and α_3 that satisfy $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $(x, m) = \alpha_1 V^1 + \alpha_2 V^2 + \alpha_3 V^3$.

Let

$$\alpha_1 = \frac{c_1 + m_1}{c_1}$$

$$\alpha_2 = \frac{a_1((a_2x_2 + m_2) - (a_2x_1 + m_1))}{2c_1(a_1 - a_2)}$$

$$\alpha_3 = \frac{a_2((a_1x_1 + m_1) - (a_1x_2 + m_2))}{2c_1(a_1 - a_2)}$$

$\alpha_1 \geq 0$ as from Inequality (2), $m_1 \geq -c_1$. $(a_2x_2 + m_2) - (a_2x_1 + m_1) \geq 0$, by Inequality (5) and the definition of envy free. $a_1 - a_2 \geq 0$, Inequality (3). Therefore $\alpha_2 \geq 0$. Indeed $(a_1x_1 + m_1) - (a_1x_2 + m_2) \geq 0$ as (x, m) is envy free and specifically Inequality (7) and so $\alpha_3 \geq 0$.

Using algebra, we can verify that

$$\alpha_1 + \alpha_2 + \alpha_3 += \cdots$$

$$= \frac{c_1 + m_1}{c_1} + \frac{1}{2} * \frac{a_1\left((a_2x_2 + m_2) - (a_2x_1 + m_1)\right)}{c_1(a_1 - a_2)} + \frac{1}{2} * \frac{a_2((a_1x_1 + m_1) - (a_1x_2 + m_2))}{c_1(a_1 - a_2)}$$

$$= \frac{c_1 + m_1}{c_1} + \frac{1}{2}\left(\frac{-2(a_1 - a_2)m_1}{c_1(a_1 - a_2)}\right)$$

$$= \frac{c_1 + m_1}{c_1} - \frac{m_1}{c_1}$$

$$= 1$$

What is left to show is that $\alpha_1 V^1 + \alpha_2 V^2 + \alpha_3 V^3 = (x, m)$

$$\frac{c_1 + m_1}{c_1}\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right) + \frac{a_1((a_2x_2 + m_2) - (a_2x_1 + m_1))}{2c_1(a_1 - a_2)}\left(\frac{1}{2} + \frac{c_1}{a_1}, \frac{1}{2} - \frac{c_1}{a_1}, -c_1, c_1\right) + \frac{a_2\left((a_1x_1 + m_1) - (a_1x_2 + m_2)\right)}{2c_1(a_1 - a_2)}\left(\frac{1}{2} + \frac{c_1}{a_2}, \frac{1}{2} - \frac{c_1}{a_2}, -c_1, c_1\right) = \cdots$$

In order for this to be true we must determine the four components of this equation

$$\frac{c_1 + m_1}{2c_1} + \left(\frac{1}{2} + \frac{c_1}{a_1}\right)\frac{a_1((a_2x_2 + m_2) - (a_2x_1 + m_1))}{2c_1(a_1 - a_2)} + \left(\frac{1}{2} + \frac{c_1}{a_2}\right)\frac{a_2\left((a_1x_1 + m_1) - (a_1x_2 + m_2)\right)}{2c_1(a_1 - a_2)} =$$

$$= \frac{c_1 + m_1}{2c_1} + \frac{(a_1 - a_2)x_1 - (a_1 - a_2)(1 - x_1) - (a_1 - a_2)m_1}{2c_1(a_1 - a_2)}$$

$$= \frac{c_1 + m_1}{2c_1} + \frac{2c_1x_1 - m_1 + c_1}{2c_1}$$

$$= x_1$$

$$\frac{c_1+m_1}{2c_1}+\left(\frac{1}{2}-\frac{c_1}{a_1}\right)\frac{a_1((a_2x_2+m_2)-(a_2x_1+m_1))}{2c_1(a_1-a_2)}+\left(\frac{1}{2}-\frac{c_1}{a_2}\right)\frac{a_2\big((a_1x_1+m_1)-(a_1x_2+m_2)\big)}{2c_1(a_1-a_2)}=$$

$$=\frac{c_1+m_1}{2c_1}+\frac{(a_1-a_2)x_2-(a_1-a_2)(1-x_2)-(a_1-a_2)m_1}{2c_1(a_1-a_2)}$$

$$=\frac{c_1+m_1}{2c_1}+\frac{2c_1x_2-m_1+c_1}{2c_1}$$

$$=x_2$$

$$(c_1)\frac{a_1\big((a_2x_1+m_1)-(a_2x_2+m_2)\big)}{2c_1(a_1-a_2)}+(c_1)\frac{a_2\big((a_1x_2+m_2)-(a_1x_1+m_1)\big)}{2c_1(a_1-a_2)}=$$

$$=\frac{2a_1c_1m_1-2a_2c_1m_1}{2c_1(a_1-a_2)}$$

$$=m_1$$

$$(c_1)\frac{a_1((a_2x_2+m_2)-(a_2x_1+m_1))}{2c_1(a_1-a_2)}+(c_1)\frac{a_2\big((a_1x_1+m_1)-(a_1x_2+m_2)\big)}{2c_1(a_1-a_2)}$$

$$==\frac{2a_1c_1m_2-2a_2c_1m_2}{2c_1(a_1-a_2)}$$

$$=m_2$$

Indeed then $\alpha_1V^1+\alpha_2V^2+\alpha_3V^3=(x,m)=\ (x_1,x_2,m_1,m_2)$

So if (x,m) is an envy free allocation it can be expressed as a convex combination of V^1 ,V^2,V^3

Claim 3.3: Suppose $0\le c_1\le\frac{a_2}{2}$. The allocations $V^1,V^2,$ and V^3 as defined by Table 4 are vertices of the set of envy-free allocations.

SubSection 2.2.2: All Possible EnvyFree Allocations are Convex Combinations

$$\left(\frac{1}{2},\frac{1}{2},0,0\right),\left(\frac{1}{2}+\frac{c_1}{a_1},\frac{1}{2}-\frac{c_1}{a_1},-c_1,c_1\right)$$

These two points have already been shown to be vertices .

$$\frac{1}{2}(x_1,x_2,m_1,m_2)+\frac{1}{2}(y_1,y_2,n_1,n_2) = \left(\frac{1}{2}+\frac{c_1}{a_2},\frac{1}{2}-\frac{c_1}{a_2},-c_1,c_1\right.$$

$$\frac{1}{2}x_1+\frac{1}{2}y_1=\frac{1}{2}+\frac{c_1}{a_2}$$

$$\frac{1}{2}x_2+\frac{1}{2}y_2 = \frac{1}{2}-\frac{c_1}{a_2}$$

$$\frac{1}{2}m_1+\frac{1}{2}n_1=-c_1$$

$$\frac{1}{2}m_2+\frac{1}{2}n_2=c_1$$

$$\left(\frac{1}{2}+\frac{c_1}{a_2},\frac{1}{2}-\frac{c_1}{a_2},-c_1,c_1\right)$$

Let's assume that $m_1 \geq \frac{-c_1}{q}$ where > 1 . Solving the third equality we get that $n_1 = -2c_1+\frac{c_1}{q}$ where $n_1 \leq -c_1$. Therefore, if m_1 does not equal $-c_1$ Inequality (2) is violated. Hence $m_1 = n_1 = -c_1$. Subsequently by Equation (2), $m_2 = n_2 = c_1$. Knowing this, we could instead write the first equation as $\frac{1}{2}x_1+\frac{1}{2}y_1=\frac{1}{2}-\frac{m_1}{a_2}$. From Inequality (6) , $x_1 \leq \frac{1}{2}-\frac{m_1}{a_2}$, therefore if $x_1 < \frac{1}{2}-\frac{m_1}{a_2}$ or $y_1 < \frac{1}{2}-\frac{m_1}{a_2}$ it would require that $y_1 > \frac{1}{2}-\frac{m_1}{a_2}$ or $x_1 > \frac{1}{2}-\frac{m_1}{a_2}$ respectively which is a violation of the Inequality. Therefore, $x_1 = y_1 = \frac{1}{2}+\frac{c_1}{a_1}$. From Equation (1) then, $x_2 = y_2 = 1-\left(\frac{1}{2}+\frac{c_1}{a_2}\right) = \frac{1}{2}-\frac{c_1}{a_2}$. Following from these results $\left(\frac{1}{2}+\frac{c_1}{a_2},\frac{1}{2}-\frac{c_1}{a_2},-c_1,c_1\right)$ must be a vertex.

As we have now fully characterized the set of envy-free allocations for any 2-player game with one homogeneous and divisible object we now will compare this set to the possible efficient allocations within the feasible set.

For an allocation to be efficient certain constraints must be met.

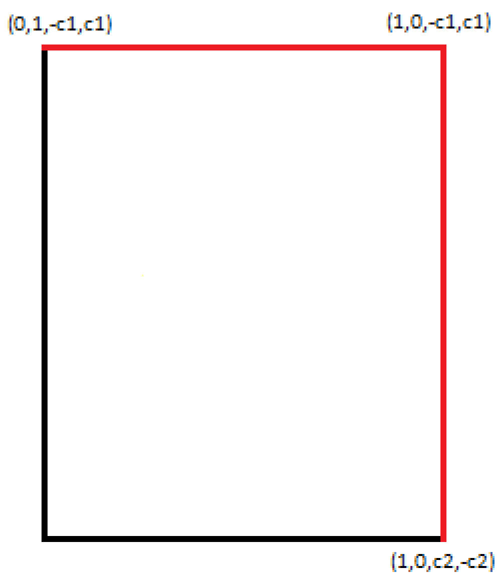
$$x_1+x_2=1$$

The unique property of efficiency is that if an allocation (x, m) is efficient then there is no other feasible allocation, (y, n), which satisfies the following,

$$u_{1,y} \geq u_{1,x} \text{ and } u_{2,y} \geq u_{2,x}$$

SubSection 2.2.2: All Possible EnvyFree Allocations are Convex Combinations

The figure below is a representation of feasible and efficient solutions in Allocation space of a 2-Player game and $c_1 \geq a_1$



Viewing this in utility space

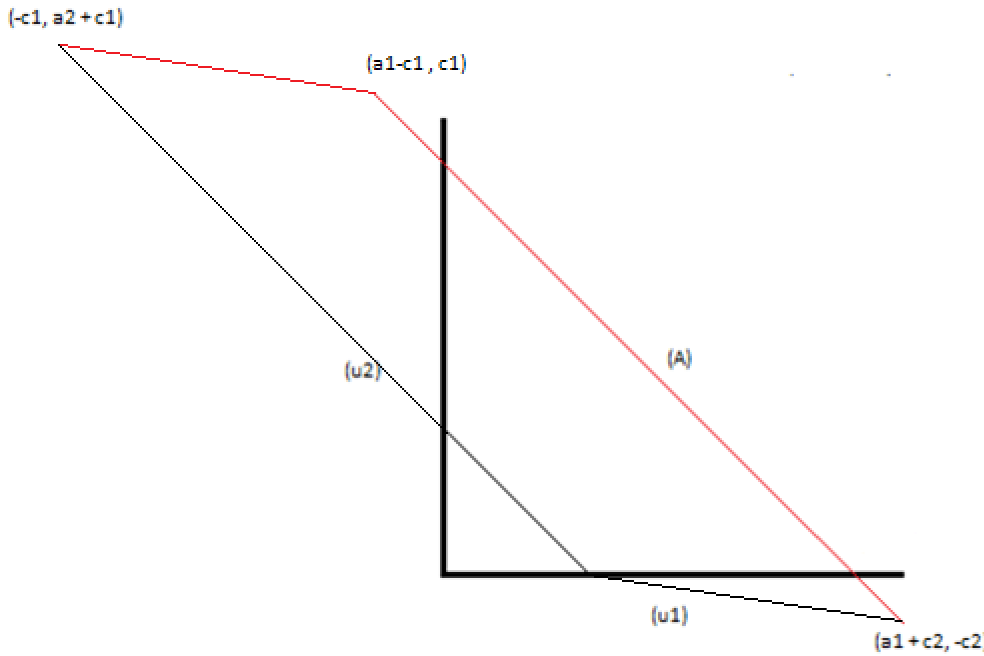


Figure ()

The three vertices, as we will analyze them in allocation space, are $V^1 = (1,0,c_2,-c_2)$, $V^2 = (1,0,-c_1,c_1)$, and $V^3 = (0,1,-c_1,c_1)$.

	(x_1,x_2,m_1,m_2)
V^1	$(1,0,c_2,-c_2)$
V^2	$(1,0,-c_1,c_1)$
V^3	$(0,1,-c_1,c_1)$

As the set of efficient allocations are the boundary of the set of feasible allocations, in order to determine if an envy free allocation (x,m) is a convex combination of these vertices then we must look at two cases. If (x,m) is a convex combination of V^1 and V^2 or if (x,m) is a convex combination of V^2 and V^3.

Claim 2.1a: If (x,m) is a convex combination of V^1 and V^2 as defined by the table in Section 2, then $(x,m)$is an efficient allocation.

We must verify that if (x, m) is a convex combination of $V^1$and V^2, then (x, m) is an efficient allocation. Suppose α_1 and α_2 are nonnegative numbers satisfying $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and

That is

$$\alpha_1 V^1 + \alpha_2 V^2 = \cdots$$

$$= \alpha_1(1, 0, \ c_2, -c_2) + \alpha_2(1, 0, -c_1, c_1)$$

Let $(x_1, \ x_2, \ m_1, \ m_2)$ be the allocation that this is equal to.

Therefore

$x_1 = \alpha_1 + \alpha_2$

$= 1$

$x_2 = 0$

$m_1 = c_2\alpha_1 - c_1\alpha_2$

$m_2 = -c_2\alpha_1 + c_1\alpha_2$

In order for (x, m) to be efficient it must first satisfy that there is no other allocation such that $u_{i,y} > u_{i,x}$ for some player i and $u_{i,y} \geq u_{i,x}$ for the rest.

Let (y, n) be an allocation where $u_{i,y} > u_{i,x}$ for some player i and $u_{i,y} \geq u_{i,x}$ for the rest.

If $u_{1,y} > u_{1,x}$ holds then it would require that Player One transfer ε of the good in return for more than $a_1\varepsilon$ monetarily. This is a contradiction as Player Two views his increase in utility from the good as $a_2\varepsilon < a_1\varepsilon$ and thus it is impossible for him/her to transfer an amount of money such that $u_{1,y} > u_{1,x}$ while maintaining $u_{2,y} \geq u_{2,x}$.

Instead assume $u_{2,y} > u_{2,x}$. Again, in this case the same argument can be made. As Player Two has none of the good it is necessary for Player One to transfer ε of the good in return for at least $a_1\varepsilon$ monetarily. The contradiction again is that $a_2\varepsilon < a_1\varepsilon$ so Player Two cannot transfer an amount to maintain Player One's utility while increasing his/her own.

It is also necessary to show that (x, m) satisfies equations (1) and (2).

$$x_1 + x_2 = 1$$

As defined above, $x_1 = 1$ and $x_2 = 0$ as well as $x_1 \geq 0$ and $x_2 \geq 0$, thus the equality holds.

$$c_2\alpha_1 - c_1\alpha_2 + -c_2\alpha_1 + c_1\alpha_2 = 0$$

Again, through simple algebra it is clear that the first equality holds.

$$\frac{c_2\alpha_1}{1-\alpha_2} \geq -c_1$$

From constraint (3) and the convex combination definiton we know $\frac{c_2\alpha_1}{1-\alpha_2} \geq 0$ and $-c_1 \leq 0$ therefore $m_1 \geq -c_1$

$$\frac{c_1\alpha_2}{1-\alpha_1} \geq -c_2$$

From constraint (3) and the convex combination definiton we know $\frac{c_1\alpha_2}{1-\alpha_1} \geq 0$ and $-c_1 \leq 0$ therefore $m_2 \geq -c_1$.

Thus, If (x, m) is a convex combination of V^1 and V^2 then (x, m) is envy free.

Claim 2.1b: If (x, m) is a convex combination of V^2 and V^3 as defined by the table in Section 2, then $(x, m)$is an efficient allocation.

That is

$$\alpha_1 V^2 + \alpha_2 V^3 = \cdots$$

$$= \alpha_1(1,0,-c_1,c_1) + \alpha_2(0,1,-c_1,c_1)$$

Let $(x_1,\ x_2,\ m_1,\ m_2)$ be the allocation that this is equal to.

Therefore

$x_1 = \alpha_1$

$x_2 = \alpha_2$

$m_1 = -c_1(\alpha_1 + \alpha_2)$

$= -c_1$

$m_2 = \alpha_1 c_1 + \alpha_2 c_1$

$= c_1$

In every possible combination of V^2 and V^3 we know that Player One is at his budget constraint.

We must also show constraint (1) and (2) are satisfied, that is the allocation is feasible.

$$x_1 + x_2 = \alpha_1 + \alpha_2$$

$$= 1 \qquad \text{by (definition of convex combination)}$$

$$m_1 + m_2 = -c_1 + c_1$$

$$= 0$$

Thus, these two constraints are satisfied as well.

Let (y, n) such that $u_{i,y} > u_{i,x}$ for some player *i* and $u_{i,y} \geq u_{i,x}$ for all the other players.

If $u_{1,y} > u_{1,x}$ then Player Two must receive a portion of the good in return for monetary compensation. As stated above Player One is at his budget constraint in every case therefore this is a contradiction. Instead assume $u_{2,y} > u_{2,x}$. Player Two cannot receive any monetary compensation from Player One thus Player One must transfer some of the good in return for some monetary compensation from Player Two. As Player Two has only part of the good it is necessary for Player One to transfer ε of the good in return for at least a_1ε monetarily. The contradiction again is that $a_2\varepsilon < a_1\varepsilon$ so Player Two cannot transfer an amount to maintain Player One's utility while increasing his/her own.

Claim 2.2. If (x, m) is an envy-free allocation, then (x, m) is a convex combination of V^1 and V^2 as defined by table above.

We verify that if (x, m) is an envy free allocation, then (x, m) is a convex combination of V^1 and V^2. We show that there exist nonnegative α_1 and α_2 that satisfy $\alpha_1 + \alpha_2 = 1$ and $(x, m) = \alpha_1 V^1 + \alpha_2 V^2$.

Let

$$\alpha_1 = \frac{m_1 + c_1 x_1}{c_1 + c_2}$$

$$\alpha_2 = \frac{c_2 x_1 - m_1}{c_1 + c_2}$$

As both V^1 and V^2 are allocations in which $x_1 = 1$ any convex combination of these vertices will have $x_1 = 1$ as well . In order for $m_1 + c_1 x_1 \geq 0$ it is necessary for $x_1 \geq -\frac{m_1}{c_1}$. From constraint (2) we know $0 \leq -\frac{m_1}{c_1} \leq 1$ therefore the inequality $x_1 \geq -\frac{m_1}{c_1}$ holds. $c_1 + c_2 \geq 0$ by constraint (4). So $\alpha_1 \geq 0$. For $\alpha_2 \geq 0$ then $x_1 \geq \frac{m_1}{c_2}$ must hold. Since $m_1 \leq 0$, by constraint (2), and $x_1 \geq 0$, by constraint (1), we know $x_1 \geq \frac{m_1}{c_2}$ holds. Therefore $\alpha_2 \geq 0$.

We need to verify that $\alpha_1 + \alpha_2 = 1$.

That is

$$\alpha_1 + \alpha_2 = \cdots$$

$$= \frac{m_1 + c_1 x_1}{c_1 + c_2} + \frac{c_2 x_1 - m_1}{c_1 + c_2}$$

$$= \frac{c_1 x_1 + c_2 x_1}{c_1 + c_2}$$

$$= x_1$$

$$= 1$$ by definition of convex combinations

We must show that $\alpha_1 V^1 + \alpha_2 V^2 = (x, m)$

$$\alpha_1 V^1 + \alpha_2 V^2 = \frac{m_1 + c_1 x_1}{c_1 + c_2}(1, 0, c_2, -c_2) + \frac{c_2 x_1 - m_1}{c_1 + c_2}(1,0, -c_1, c_1)$$

The four components of this equality are

$$x_1 = \frac{m_1 + c_1 x_1}{c_1 + c_2} + \frac{c_2 x_1 - m_1}{c_1 + c_2} = 1$$ by the computation of $\alpha_1 + \alpha_2$

$$x_2 = 0$$

$$m_1 = \frac{c_2 m_1 + c_1 c_2 x_1}{c_1 + c_2} + \frac{c_1 m_1 - c_2 c_1 x_1}{c_1 + c_2}$$

$$= \frac{c_2 m_1 + c_1 m_1}{c_1 + c_2}$$

$$= m_1$$

$$m_2 = \frac{-c_2 m_1 - c_1 c_2 x_1}{c_1 + c_2} + \frac{c_2 c_1 x_1 + c_1 m_1}{c_1 + c_2}$$

$$= \frac{-c_2 m_1 - c_1 m_1}{c_1 + c_2}$$

$$= -m_1$$

Thus, $\alpha_1 V^1 + \alpha_2 V^2 = (x, m) = (1,0, m_1, -m_1)$

So, if (x, m) is an envy free allocation, then (x, m) is a convex combination of V^1 and V^2.

Claim 2.2b. If (x, m) is an envy-free allocation, then (x, m) is a convex combination of V^2 and V^3 as defined by table above.

We verify that if (x, m) is an envy free allocation, then (x, m) is a convex combination of V^2 and V^3. We show that there exist nonnegative α_1 and α_2 that satisfy $\alpha_1 + \alpha_2 = 1$ and $(x, m) = \alpha_1 V^2 + \alpha_2 V^3$.

Let

$$\alpha_1 = x_1$$

$$\alpha_2 = x_2$$

$\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ by constraint (1).

We need to verify that $\alpha_1 + \alpha_2 = 1$.

That is

$$\alpha_1 + \alpha_2 = x_1 + x_2$$

$$= 1 \qquad \text{by equation (1)}$$

We must show that $\alpha_1 V^2 + \alpha_2 V^3 = (x, m)$

That is

$$\alpha_1 V^2 + \alpha_2 V^3 = x_1(1,0,-c_1,c_1) + x_2(0,1,-c_1,c_1)$$

The four components of this equality are

$$x_1 = x_1$$

$$x_2 = x_2$$

$$m_1 = -c_1 x_1 - c_1 x_2$$

$$= -c_1 \qquad \text{by equation (1)}$$

$$m_2 = c_1 x_1 + c_1 x_2$$

$$= c_1 \qquad \text{by equation (1)}$$

Therefore $\alpha_1 V^2 + \alpha_2 V^3 = (x, m) = (x_1, x_2, -c_1, c_1)$

So, if (x, m) is an envy free allocation, then (x, m) is a convex combination of V^2 and V^3.

Claim 2.3 : The allocations V^1, V^2, and V^3 as defined by the table in Section 2 are vertices of the set of efficient allocations

We now verify that V^1, V^2 and V^3 are vertices of the efficient allocations. We will do so by showing that if the average of any two points in the set of envy free allocations equals this allocation then these two points are in fact the same, or, the allocation itself.

Suppose(x_1, x_2, m_1, m_2) and (y_1, y_2, n_1, n_2) are efficient and

$$\frac{1}{2}(x_1, x_2, m_1, m_2) + \frac{1}{2}(y_1, y_2, n_1, n_2) = (1,0, c_2, -c_2\ )$$

That is

$$\frac{1}{2}x_1 + \frac{1}{2}y_1 = 1$$

$$\frac{1}{2}x_2 + \frac{1}{2}x_2 = 0$$

$$\frac{1}{2}m_1 + \frac{1}{2}n_1 = c_2$$

$$\frac{1}{2}m_2 + \frac{1}{2}n_2 = -c_2$$

Starting with the first equality, if $x_1 < 1$ or $y_1 < 1$ then it would be necessary for either $y_1 > 1$ or $x_1 > 1$, respectively, a contradiction to constraint (1). So $x_1 = y_1 = 1$.Then, $x_2 = y_2 = 0$ using equation(1).

From the fourth equality, if $m_2 > -c_2$ or $n_2 > -c_2$ it would be necessary for either $m_2 < -c_2$ or $n_2 < -c_2$, respectively, a contradiction to constraint (3). So, $m_2 = n_2 = -c_2$. Finally $m_1 = n_1 = c_2$ using equation (2) and the corresponding values of m_2 and n_2 .

Therefore V^1 is a vertex of the efficient set.

Suppose (x_1, x_2, m_1, m_2) and (y_1, y_2, n_1, n_2) are efficient and

$$\frac{1}{2}(x_1, x_2, m_1, m_2) + \frac{1}{2}(y_1, y_2, n_1, n_2) = (1,0, -c_1, c_1\ )$$

That is

$$\frac{1}{2}x_1 + \frac{1}{2}y_1 = 1$$

$$\frac{1}{2}x_2 + \frac{1}{2}x_2 = 0$$

$$\frac{1}{2}m_1 + \frac{1}{2}n_1 = -c_1$$

$$\frac{1}{2}m_2 + \frac{1}{2}n_2 = c_1$$

Starting with the first equality, if $x_1 < 1$ or $y_1 < 1$ then it would be necessary for either $y_1 > 1$ or $x_1 > 1$, respectively, a contradiction to constraint (1). So $x_1 = y_1 = 1$.Then, $x_2 = y_2 = 0$ using equation(1). From the third equality, if $m_1 > -c_1$ or $n_1 > -c_1$ it would be necessary for either $m_1 < -c_1$ or $n_1 < -c_1$, respectively, a contradiction to constraint (3). So, $m_1 = n_1 = -c_1$. Finally $m_2 = n_2 = c_1$ using equation (2) and the corresponding values of m_2 and n_2 .

Therefore, V^2 is a vertex.

Suppose (x_1, x_2, m_1, m_2) and (y_1, y_2, n_1, n_2) are efficient and

$$\frac{1}{2}(x_1, x_2, m_1, m_2) + \frac{1}{2}(y_1, y_2, n_1, n_2) = (0,1, -c_1, c_1)$$

That is

$$\frac{1}{2}x_1 + \frac{1}{2}y_1 = 0$$

$$\frac{1}{2}x_2 + \frac{1}{2}x_2 = 1$$

$$\frac{1}{2}m_1 + \frac{1}{2}n_1 = -c_1$$

$$\frac{1}{2}m_2 + \frac{1}{2}n_2 = c_1$$

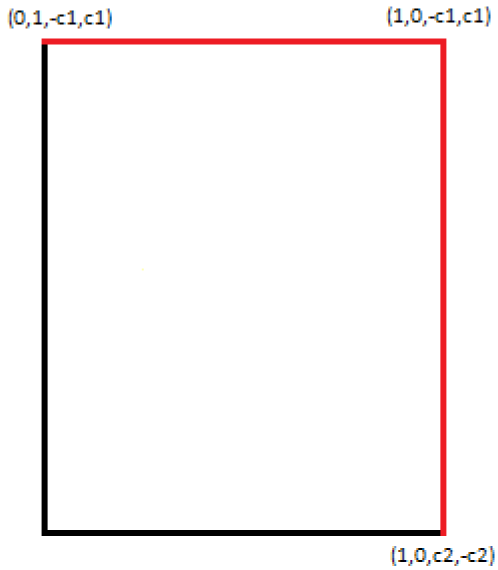
Starting with the second equality, if $x_2 < 1$ or $y_2 < 1$ then it would be necessary for either $y_2 > 1$ or $x_2 > 1$, respectively, a contradiction to constraint (1). So $x_2 = y_2 = 1$.Then, $x_1 = y_1 = 0$ using equation(1). From the third equality, if $m_1 > -c_1$ or $n_1 > -c_1$ it would be necessary for either $m_1 < -c_1$ or $n_1 < -c_1$, respectively, a contradiction to constraint (3). So, $m_1 = n_1 = -c_1$. Finally $m_2 = n_2 = c_1$ using equation (2) and the corresponding values of m_2 and n_2 .

Therefore, V^3 is a vertex.

SubSection 2.2.2: All Possible EnvyFree Allocations are Convex Combinations

What must be shown for both of the cases where (x, m) is a convex combination of either V^1 and V^2 or V^2 and V^3 is that if (x, m) is not a convex combination of these vertices then it is not efficient.

It makes the most sense to view this geometrically in allocation space.



The set of efficient allocations is the borders of the feasible allocations where $x_1 = 1$ or where $m_1 = -c_1$. If there were an allocation (y, n) that did not lie on this border then we know Player One is not receiving all of the good and is not as his budget constraint so it would be possible for Player Two to transfer ε of the good to Player One, where $\varepsilon > 0$.

We want to show that this new allocation increases either Player One or Player Two's utility and leaves the other as least as high as it previously was.

The transformed utility for Player One is equivalent to

$$u_{1,y} = (x_1 + \varepsilon)a_1 - m_1 - \varepsilon a_1$$

$$= a_1 x_1 - m_1$$

$$= u_{1,x}$$

We want to show that this new allocation increases either Player One or Player Two's utility and leaves the other as least as high as it previously was.

The transformed utility for Player Two is equivalent to

$$u_{2,y=}(x_2 - \varepsilon)a_2 + m_1 + \varepsilon a_1$$

$$(x_2 - \varepsilon)a_2 + m_1 + \varepsilon a_1 > x_2 a_2 + m_1$$

Since $a_1 > \; a_2$, by constraint (3), we know $\varepsilon a_1 > \varepsilon a_2$ therefore $u_{2,y} > u_{2,x}$. This means that the allocation (x, m) is not efficient as there exists an allocation (x, m) in which at least one player is better off while the other remains at least the same.

Thus, if (x, m) is not a convex combination of either V^1 and V^2 or V^2 and V^3 then (x, m) is not efficient.